Radiation reaction from QED

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We derive radiation reaction directly from the S-matrix of QED. We identify the diagrams and processes contributing to recoil effects in the average momentum of a scattered electron and, in an explicit example, find exact agreement with classical electrodynamics in the limit $\hbar \to 0$. Our results unify a number of disparate, previously reported approaches.

Understanding radiation reaction presents one of the oldest problems in electrodynamics, and has recently seen a renewal of interest due to the potential impact, and detection, of recoil effects in high-intensity laser-matter interactions [1, 2]. The years have seen many proposals for equations which describe a classical, radiating particle and avoid the problems of the Lorentz-Abraham-Dirac ('LAD') equation [3–6], but without consensus [7]. Nor is it completely understood how radiation reaction ('RR') emerges from QED [1]. For example, the photon spectrum emitted from an electron in a background field gives, in the classical limit, the radiation spectrum of a particle moving according to the Lorentz equation; no recoil effects are seen [8, 9]. This has prompted proposals for which Feynman diagrams contain RR [10], and the claim that quantum and classical electrodynamics are not compatible [11].

Given the great interest in this topic, we will here derive both quantum and classical RR directly and unambiguously from QED. In doing so we will unify several existing approaches (such as use of the in-in formalism and coherent vs incoherent emissions), and resolve the above issue with the photon spectrum. Rather than pursue an equation, see e.g. [12] for quantum stochastic corrections to LAD, we will consider scattering in QED, explain which diagrams contribute to RR, and illustrate these statements with an explicit example. All we will do is calculate (the expectation value of) the momentum of a quantum particle, and take $\hbar$ to zero to obtain the classical momentum. For our example, we consider scattering in a plane wave background modelling, say, an intense laser. With this choice, the Lorentz force can be treated without approximation, allowing a clean separation of recoil (the impact of radiation on the electron's motion) and non-recoil effects.

This paper is organised as follows. We begin by reviewing how and where RR arises classically. We then identify where RR arises in QED, stating which diagrams contribute. The electron recoil is then written down, and we take $\hbar \to 0$, finding agreement with classical predictions. Connections to other approaches are then discussed before we conclude.

Classical:– The LAD equation for a radiating particle with orbit $x^\mu(\tau)$, proper time τ, is ($c = \epsilon_0 = 1, \hbar \neq 1$)

$$m\ddot{x}_\mu = eF^{\text{ext}}_{\mu\nu}(x)\dot{x}^\nu + \frac{2}{3}\frac{e^2}{4\pi}(\dddot{x}_\mu\dot{x}_\nu - \dot{x}_\mu\dddot{x}_\nu)\dot{x}^\nu \ , \qquad (1)$$

where F^{ext} is an external field [13]. Retaining only the first term on the right hand side of (1) gives the Lorentz force equation, the second term describes RR. The runaway solutions LAD predicts can be avoided through asymptotic boundary conditions [13], performing a reduction of order to obtain the Landau Lifshitz (LL) equation [3], using an alternate equation [4–6], or (since runaways are nonperturbative in the classical electron radius $r_0 = e^2/4\pi m$) simply by expanding the orbit in r_0.

In order to compare directly with QED, it is helpful to go back to the classical equations of motion and simply solve them perturbatively (rather than eliminating the radiation field to obtain LAD), in such a way that RR effects appear as corrections to the Lorentz force. To do so, define $f := eF^{\text{ext}}$, let F be the dynamical electromagnetic field with potential A, and j the current. The classical equations of motion are then, schematically [13],

$$\ddot{x} = f\dot{x} + eF\dot{x} \ , \qquad \Box A = ej \ . \qquad (2)$$

(The external field obeys Maxwell's equations in vacuum, $\Box A_{\text{ext}} = 0$). We solve (2) perturbatively in e, expanding

$$\begin{aligned} x &= x_0 + ex_1 + \ldots \implies j = j_0 + ej_1 + \ldots \\ A &= A_0 + eA_1 + \ldots \implies F = F_0 + eF_1 + \ldots \end{aligned} \qquad (3)$$

To zeroth order, we have $\Box A_0 = 0$ and $\ddot{x}_0 = f(x_0)\dot{x}_0$, the Lorentz force equation. So, the particle moves according to the Lorentz force but, with appropriate initial conditions, $A_0 = F_0 = 0$ and there is no radiation. To first order in e, we find a *homogeneous* equation for x_1. Since initial conditions can be fulfilled by x_0, x_1 can be set to zero, and the particle's motion is unaffected. At this order we also have $\Box A_1 = j_0$, and hence a nonzero radiation field F_1 sourced by x_0, i.e. by a particle moving under the Lorentz force. Radiation is therefore created at order e, but field observables are typically of order e^2, so the lowest order radiated energy, for example, is $E^2 + B^2 \sim e^2 j_0^2$, which does not contain recoil.

At second order in e, one finds $\Box A_2 = F_2 = 0$ and

$$\ddot{x}_2 = f(x_0)\dot{x}_2 + x_2\partial f(x_0)\dot{x}_0 + F_1(x_0)\dot{x}_0 \ . \qquad (4)$$

This *inhomogeous* equation yields a nonzero x_2. The particle's orbit is corrected due to the term $F_1\dot{x}_0$, i.e. by the fact that the particle has emitted the radiation F_1; this is radiation reaction. It appears at order e^2 in the electron's motion, as expected. At third order, one finds

that $F_3 \neq 0$, sourced by the *radiating* particle, i.e. x_2 contributes to the radiation field. Hence recoil effects appear in the radiated energy first at order e^4, through the $F_1F_3 \sim j_0 j_2$ cross term.

Example:– We illustrate the above for a background plane wave depending on invariant phase $\phi := k.x$ with $k^2 = 0$ [1]. We have $k_\mu = \omega n_\mu$ for ω an inverse-length scale, say a central frequency, and we can choose coordinates such that $k.x = \omega x^+$, lightfront time. The transverse vector $a'_\perp$ $(k.a' = 0)$ is the normalised electric field, $eE_\perp \equiv \omega a'_\perp$. We consider pulses, so that $E_\perp(\phi)$ is either nonzero only in a finite ϕ-range or vanishes asymptotically, but is otherwise arbitrary. The momentum $\pi^\mu := m\dot{x}_0^\mu$ of a particle with initial momentum p_μ is, according to the Lorentz force [14, 15],

$$\pi_\mu(\phi) = p_\mu - a_\mu(\phi) + \frac{2p.a(\phi) - a^2(\phi)}{2k.p} k_\mu \ , \qquad (5)$$

and one has that $k.p$ is conserved, hence $\phi \propto \tau$. (From here on, an integral without variable is over lightfront time $\mathrm{d}\phi$, $\pi \equiv \pi(\phi)$ and $\hat{\pi} \equiv \pi(\infty)$ is the final momentum as given by the Lorentz force in (5).) We proceed to x_2^μ. LAD, LL and the equations in [4–6], while giving different results for x_2^μ within the background, all agree on the first RR correction to the *final* momentum of a particle passing through the whole pulse; this is $\delta\pi_\mu$, with

$$\delta\pi_\mu = \frac{2}{3}\frac{e^2}{4\pi}\frac{k.p}{m^4}\int a'^2\bigg(\pi_\mu - \frac{\hat{\pi}.\pi}{k.p}k_\mu\bigg) \ . \qquad (6)$$

An explanation of why all the equations agree on this asymptotic result, to this order, is as follows. The first term in (6) can also be obtained by inserting the Lorentz orbit x_0^μ into Larmor's formula for the total radiated momentum. The second term in (6), representing momentum taken from the background, then follows by momentum conservation.

Classical to quantum:– A direct, perturbative solution of the classical theory has yielded the following. To zeroth order in e, a particle is accelerated by an external field but does not emit. To first order, the emitted radiation of the particle is accounted for. To second order, the effect of this emission on the electron is accounted for. The observables at this order include the corrected electron orbit, and the emission spectrum of a particle moving *according to the Lorentz equation.* At third order, the impact of recoil on the radiation field is accounted for, and enters the field's observables at order e^4.

Where will we find such effects in QED? Consider QED with a background field. In the Furry picture, interactions between quantised fields are treated perturbatively as normal, while the background is treated as a part of the 'free' theory and, classically, effects the Lorentz force on the fermions [15–17]. (The Furry picture therefore has the usual Feynman diagram expansion but with the free fermion propagator replaced by that in the background.) Hence, perturbative QED in the Furry picture is an expansion in powers of $\alpha = e^2/(4\pi\hbar)$ beyond a 'free' theory without recoil, but with the Lorentz force. It is the quantum equivalent of the expansion used above. Thus, lowest order RR corrections should come from (the classical limit of) order e^2 Furry picture processes in QED. We confirm this below.

Radiation reaction from QED:– Consider the following simple scattering experiment. An electron is collided with a laser pulse, and its momentum is measured after exiting the beam. The classical theory, using (1) or otherwise, predicts that the electron will have a certain momentum; $\hat{\pi}+\delta\pi$ from (6) for a plane wave background. The quantum theory predicts that, repeating the experiment, a distribution of final momenta will be measured. The expectation value of this distribution can be calculated in QED; it is the expectation value of the electron momentum operator in the final state describing the scattering process. Expectation values, unlike probabilities, have natural classical limits; this is why approaches based on them, such as the in-in formalism [18], are common in studying how classical RR arises from QED [12, 19, 20]. Such approaches, though, are not as widely used in particle physics as scattering amplitudes, so it is important to understand how RR arises in the latter.

Returning to our experiment, we note that measurements are unlikely to be made within the fields of the laser, and so asymptotic results are sufficient. In this case, we can easily relate the expectation value of interest to familiar S-matrix elements and Feynman diagrams, as follows. We begin with an incoming electron state $|\, i\,\rangle$, evolve it in time through a background field, and calculate the expectation value of the electron momentum operator, Π_μ, in the evolved state. (In [21], the position shift of a scattered particle was compared between quantum and classical theories. Here we compare momenta, which are the natural variables in scattering calculations.) If S is the S-matrix, the expectation value is $\langle\Pi_\mu\rangle = \langle\, i\,|S^\dagger\Pi_\mu S|\, i\,\rangle$. Inserting a complete set of asymptotic out states $|\, f\,\rangle$ with $\Pi|\, f\,\rangle = p^f|\, f\,\rangle$, we can write the expectation value in terms of S-matrix elements $S_{fi} = \langle\, f\,|S|\, i\,\rangle$ as

$$\langle\Pi_\mu\rangle = \sum_f p_\mu^f \ |S_{fi}|^2 \ , \qquad (7)$$

neatly relating the in-in and scattering formalisms. The sum in (7) runs over all Fock-sectors of free particle states, and can be interpreted as a sum of amplitudes for the processes $\mathrm{e}^- \to \mathrm{e}^- +$ anything. Each differential probability $|S_{fi}|^2$ in (7) has a Furry-Feynman diagram expansion in powers of e. The zeroth order contribution to (7) comes from the one-electron sector, i.e. the process $\mathrm{e}^- \to \mathrm{e}^-$ without emission, at tree level. Classically, it describes only the effect of the Lorentz force. The next contributions to $\langle\Pi_\mu\rangle$ come at order e^2 from two processes, see Fig. 1. First, photon emission at tree level,

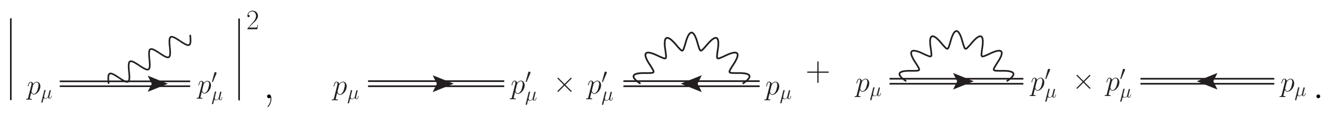

FIG. 1. Feynman diagrams contributing to radiation reaction at lowest order. (A double line is the fermion propagator in a background field.)

mod-squared. Second, the cross term of $e^- \to e^-$ to one loop, describing the self-field of the electron. Together, these processes give a nonzero contribution to the final electron momentum and $\langle\Pi_\mu\rangle$; this is quantum radiation reaction to lowest order. The loop cannot be neglected. First, simply for consistency; one cannot keep one order e^2 contribution and drop another. Second, without the loop, $\langle\Pi_\mu\rangle$ would be infra-red divergent in general. It is a standard result that only the inclusive sum of the loop and emission processes is IR finite and contributes to observables [22]. Such sums are automatically included in (7); this is an advantage of considering $\langle\Pi\rangle$ rather than individual diagrams.

Example:– We now illustrate the above for a plane wave background, take $\hbar \to 0$ and recover the classical result (6). As above, the zeroth order contribution to $\langle\Pi_\mu\rangle$ comes from tree-level scattering without emission, and yields $\langle\Pi_\mu\rangle = \hat{\pi}_\mu$ [23], the Lorentz force result. We proceed to order e^2. Photon emission $e^-(p) \to e^-(p') + \gamma(k')$ in a plane wave, called nonlinear Compton scattering, yields an outgoing momentum $p^f = p'$ with

$$p'_\mu = \hat{\pi}_\mu - k'_\mu + \frac{k'.\hat{\pi}}{k.(p-k')} k_\mu \; , \qquad (8)$$

and this should be inserted into the integrated probability of emission to obtain its contribution to $\langle\Pi_\mu\rangle$, see [15]. Using similar methods, one finds that the loop in Fig. 1 leads to the same contribution as emission, except that the final momentum $p^f = -\hat{\pi}$ rather than (8). The total contribution to the final electron momentum at order e^2, call it $\delta\langle\Pi_\mu\rangle$, is then ($\pi_j = \pi(\phi_j)$, $a_j = a(\phi_j)$)

$$\delta\langle\Pi_\mu\rangle = \frac{e^2}{4\pi\hbar} \int\limits_0^{p_-} \frac{\mathrm{d}k'_-}{k'_-} \int \frac{\mathrm{d}^2 k'_\perp}{(2\pi)^2} \frac{k.p'}{k.p} \left(\hat{\pi}_\mu - p'_\mu\right) \int \mathrm{d}\phi_1 \mathrm{d}\phi_2$$

$$\exp\left[\frac{i}{\hbar} \int\limits_{\phi_1}^{\phi_2} \frac{k'.\pi}{k.p'}\right] \partial_2\partial_1 \left(\frac{m^2 - g(t)[a_2 - a_1]^2}{k'.\pi_2 k'.\pi_1}\right), \qquad (9)$$

and spin effects appear in

$$g(t) = \frac{1 + (1+t)^2}{4(1+t)} \text{ where } t = \frac{k.k'}{k.p'} \; . \qquad (10)$$

The result (9) is finite, nonzero and has support on the difference between the electron's momentum following photon emission, (8), and the Lorentz force (no recoil) result, $\hat{\pi}$ [24]. It is due to combined photon emission and self-energy effects. This is quantum radiation reaction in a plane wave background.

It is now known that momentum integrals such as those in (9) can be performed analytically and exactly [25], opening the way for efficient calculations of cross sections in plane waves. Our objective here, though, is to take the classical limit of $\delta\langle\Pi_\mu\rangle$. To do so, we note that in a plane wave, it is the emitted photon's longitudinal momentum $k.k'$ which is important. This 'breaks the symmetry' of motion associated with the Lorentz force [26], namely the conservation of $k.p$. This is a sign of RR, as is seen explicitly when one solves LL in a plane wave [27]. Hence, we will rewrite (9) to highlight its dependence on $k.k'$. Let $k'_\perp = (k.k'/k.p) r_\perp$ and change variables $k'_\perp \to r_\perp$, and $k'_- \to t = k.k'/k.p'$, the ratio of longitudinal momenta in the final state. Finally, noting that photon momentum has no classical analogue, we rescale $t \to \hbar t$, which corresponds to using wavenumber rather than momentum. This makes all dependencies on $\hbar$ manifest. The result is that (9) is equal to ($r_- = p_-$ since k' is on-shell)

$$\delta\langle\Pi_\mu\rangle = \frac{e^2}{4\pi} \int\limits_0^\infty \frac{\mathrm{d}t}{(1+\hbar t)^2} \int \frac{\mathrm{d}^2 r_\perp}{(2\pi)^2} \left[\frac{r_\mu}{1+\hbar t} - \frac{r.\hat{\pi}}{k.p} k_\mu\right] \qquad (11)$$

$$\int \mathrm{d}\phi_1 \mathrm{d}\phi_2 \exp\left[it \int\limits_{\phi_1}^{\phi_2} \frac{r.\pi}{k.p}\right] \partial_2\partial_1 \left(\frac{m^2 - g(\hbar t)[a_2 - a_1]^2}{r.\pi_2 r.\pi_1}\right) ,$$

where t is given by the following simple combination of final (scattered) momenta,

$$\hbar t = \frac{k.k'}{k.p'} = \frac{k.k'}{k.p - k.k'} \; . \qquad (12)$$

$\delta\langle\Pi_\mu\rangle$ depends on $\hbar$ only through the combination $1 + \hbar t$, so the classical limit $\hbar \to 0$ can be taken simply by expanding these factors in (11). From (12), this is equivalent to assuming that $k.k' \ll k.p'$ (implying $k.k' \ll k.p$), i.e. that the momentum carried away by the photon is small compared to that of the electron. The classical limit therefore corresponds to neglecting quantum effects associated to emission of high energy photons. It remains to evaluate the t and $r_\perp$ integrals at $\hbar = 0$. Noting that $\delta\langle\Pi_\mu\rangle$ is real and symmetric in $\phi_1 \leftrightarrow \phi_2$, the t integral can be symmetrised and immediately gives a delta function for the two ϕ-integrals,

$$\frac{1}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}t \cos\left(t \int\limits_1^2 \frac{r.\pi}{k.p}\right) = \pi \frac{k.p}{r.\pi(\phi_2)} \delta(\phi_2 - \phi_1) \; . \qquad (13)$$

This result means that while quantum RR (9) is coherent in (lightfront) time, coming from an S-matrix element squared, classical RR is incoherent; (13) is the statement that interference terms drop out and only the incoherent piece of the ϕ–integrals in (9) survives as $\hbar \to 0$. We therefore see a simple aspect of decoherence, which is intrinsic to quantum-to-classical transitions [28]. Now, using (13) to eliminate one of the ϕ-integrals, the remaining $r_\perp$ integrals become elementary, and we obtain

$$\lim_{\hbar\to 0} \delta\langle\Pi_\mu\rangle = \frac{2}{3}\frac{e^2}{4\pi}\frac{k.p}{m^4}\int a'^2\bigg(\pi_\mu - \frac{\hat{\pi}.\pi}{k.p}k_\mu\bigg) = \delta\pi_\mu \ , \quad (14)$$

recovering the classical result (6). We have now obtained classical radiation reaction directly from QED [29].

Discussion:– Related calculations appear in [30], which partially recovers classical RR in a high energy approximation, and in [11], which concludes that QED supports the classical equation in [6]. (We will address this elsewhere.) In perturbation theory, $k.\delta\pi$ from (6) can be obtained from ordinary Compton scattering if the incoming photon flux is equal to the plane wave's energy density a'^2 [14]. That paper finds that quantum and LAD results differ for large recoil. This is explicit in our approach; accounting for larger recoil requires retaining higher order terms in the expansion of (11), and these terms are of higher order in $\hbar$.

We now turn to higher order RR effects. The sum in (7) is incoherent in particle number (as in [10]), but each process is coherent in time. Quantum RR thus retains some coherence, in general. n-photon emission contributes to RR at order e^{2n}, but so do all other processes of the same order, e.g. pair production, loop corrections, counterterms and so on. These cannot be neglected in general. First, for consistency. Second, they remove UV and IR divergences. Third, unitarity can be violated without them. It was noted in [10] that photon emission probabilities in plane waves can exceed unity. Such problems do not appear here; the terms which violate unitarity are removed by consistent inclusion of, e.g. the above loop, see also [25].

Further, our calculation shows that quantum processes can also play an important role in the classical limit. We find that the loop does not contribute directly to classical RR (which is natural) because it cancels against a term coming from the emission diagram; however, the cancelling terms are $\mathcal{O}(1/\hbar)$ and would otherwise diverge in the classical limit. Hence it seems that the loop is needed for both quantum and classical RR. See [31] more on $\hbar$ and the classical limit.

Finally, it is now clear (even from our classical discussion) why one-photon emission spectra do not exhibit RR [8, 9]; at order e^2, such effects are only visible in the electron sector. The reason is that the probability of emitting a photon at all is already order e^2, so one might say that a single emitted photon does not know that the electron it leaves behind has recoiled. RR effects appear in the *photon* spectrum at higher order, following, e.g., multiple emissions. For the electron, there is no essential difference between single and multi-photon emissions. Both contribute to RR.

Conclusions:– Classical radiation reaction emerges directly from the $\hbar \to 0$ limit of QED. We have identified, in general, the contributing processes and diagrams; to lowest order, recoil effects in the electron momentum come from one-photon emission and one-loop self-energy.

In an explicit example, we have seen how the recoil of an accelerated, emitting electron recovers classical radiation reaction in the low-energy limit. The advantage of using a plane wave as the background in this example is that no further approximations need be made; one simply takes $\hbar \to 0$, thus eliminating potential ambiguities. Our calculation allows for a clear identification of recoil effects and unites the usual S-matrix approach to QED with 'in-in' methods and the calculation of expectation values.

Asymptotic results following from the S-matrix, as used here, are sufficient for comparison with experiments aiming to observe radiation reaction, since scattering products will be measured far from interaction volumes. (Previous position shift calculations which find agreement between quantum and classical theories are also for 'asymptotic' times [21].) Nevertheless, the extension of our calculation to finite time and higher orders (it will be interesting to explore renormalisation effects here) can distinguish between, and therefore rule out, different classical equations which predict different motion within the background field. Our results will appear soon.

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